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Presentation Outline

- Please ignore spelling errors
- Please interupt me to ask questions or clarifications

Stanley (old)

$(-1)^n \chi_G(-1) = \#$ of acyclic orientations

Zaslavsky

definitions

examples of cycles in signed directed graphs

generalization of Stanley to signed graphs

Stanley's theorem

From Dr. Chmutov's Lecture notes:

1.3 **Stanley's theorem.** *For a graph G with n vertices, $(-1)^n\chi_G(-1) = \#$ of acyclic orientations of G.*

Steps to proving this:

1.1 every proper coloring has an associated orientation

↓

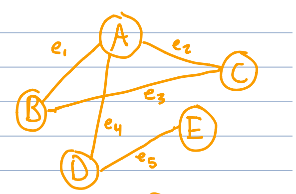
1.2 $\bar{\chi}(\lambda) = (-1)^p \chi(-\lambda)$

↓

1.3 goal

Definitions and variables

- G is a finite graph with out loops or multiedges
- $V = V(G)$ is the set of vertricies of graph G
- $X = X(G)$ is the set of edges of G
- $e \in X$ is an unordered pair $\{u, v\}$ of verticies where $u \neq v$
- $p = |V(G)|$ the number of verticies in G
- $q = |X(G)|$ the number of edges in G
- orientation - an assignment of a direction to each edge denoted by $u \to v$ or $v \to u$
- acyclic - an orientation of G has no directed cycles



G

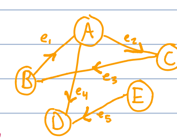
$V(G) = \{A, B, C, D, E\}$

$X(G) = \{e_1, e_2, e_3, e_4, e_5\}$

$e_1 = \{B, A\}$

$p = |V(G)| = 5$

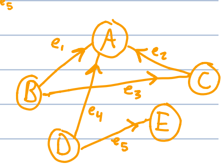
$q = |X(G)| = 5$



not acyclic

cycle ABCA

acyclic



- $\chi(\lambda)=\chi(G,\lambda)$ is the chromatic polynomial of G evaluated at λ colors

  for $\lambda \in \mathbb{N}$ $\chi(G,\lambda) = \#$ of proper colorings in λ colors
- K is any map $K: V \to \{1,2,\dots \lambda\}$ i.e. K is a coloring
- $\mathcal{O}$ will be a certain orientation
- $\bar{\chi}(\lambda)$ is $\chi(\lambda)$ with a different condition
- improper coloring is where an edge $\{u,v\}$ has $K(u)=K(v)$

<u>Proposition 1.1</u> - Every proper coloring has an associated orientation $\mathcal{O}$

$\chi(\lambda)$ is equal to the number of pairs $(K,\mathcal{O})$

recall K is any map $K: V \to \{1,2,\dots \lambda\}$ (a coloring)

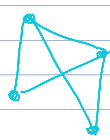
$\mathcal{O}$ is an orientation of G with the following conditions

(1) the orientation $\mathcal{O}$ is acyclic
(yes this is redundant but it helps to transition to $\bar{\chi}$)

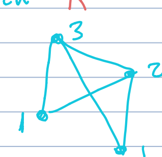
(2) If $u \to v$ in the orientation $\mathcal{O}$, then $K(u) > K(v)$

Lets do an example to understand $(K,\mathcal{O})$:

Ex 1.

Let G be  and let $\lambda = 3$ thus $K: V \to \{1,2,3\}$

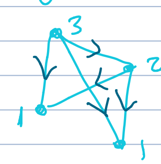
I am going to pick K



note that the orientation rule (2) does not allow improper colorings

Lets orient this graph based on the coloring

(2) If $u \to v$ in the orientation $\mathcal{O}$, then $K(u) > K(v)$



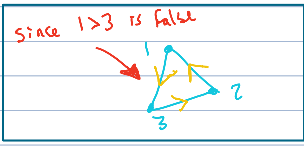
$\mathcal{O}$

Then $\chi(3) = \#$ of $(K,\mathcal{O})$

This example should give you intuition about the proof so I will breifly touch upon the proof, but after this presentation you should go back and convince yourself if you truly care about math.

proof 1.1

- note that condition (2) forces the graph to be acyclic



Assume to get a contradiction that there is a cycle

$u \to v \to w_1 \to w_2 \cdots \to u$

but then based on condition 2:

$\underline{\kappa(u) >} \kappa(v) > \kappa(w_1) \cdots \underline{> \kappa(u)}$

But This is Wrong !!!

since $\kappa(u) \ngtr \kappa(u)$

Thus condition 2 forces acyclicity

(2) If $u \to v$ in the orientation $\mathcal{O}$, then $\kappa(u) > \kappa(v)$

- note that condition (2) implies that the coloring κ is proper since

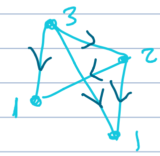
$\boxed{\text{(2) If } u \to v \text{ in the orientation } \mathcal{O}, \text{ then } \kappa(u) > \kappa(v)}$

$\kappa(u) > \kappa(v) \Rightarrow \kappa(u) \neq \kappa(v)$

Thus if you have a pair $(\kappa, \mathcal{O})$, κ must be proper

$\boxed{\text{Conversley}}$ if κ is a proper coloring then it corespods to a unique $(\kappa, \mathcal{O})$ pair:

Lets orient this graph based on the coloring



$\mathcal{O}$

Thus # of pairs $(\kappa, \mathcal{O})$ = # of κ = # of proper colorings = $\chi(\lambda)$ ∎

Now we are going to define $\overline{\chi}(\lambda)$:

$\overline{\chi}(\lambda)$ is equal to the number of pairs $(K, \mathcal{O})$

recall K is any map $K: V \to \{1, 2, \dots \lambda\}$ (a coloring)

$\mathcal{O}$ is an orientation of G with the following conditions

(1) the orientation $\mathcal{O}$ is acyclic
(not redundant anymore)

(2) If $u \to v$ in the orientation $\mathcal{O}$, then $K(u) \geq K(v)$

Theorem 1.2 $\quad \overline{\chi}(\lambda) = (-1)^p \chi(-\lambda)$ for $\lambda \in \mathbb{Z}$ where $\lambda \geq 0$ and $p = \#$ of verticies

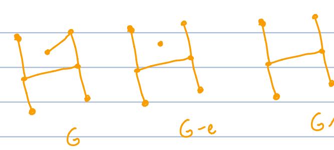
this will be proved using induction, but first some properties of $\overline{\chi}(\lambda)$:

recall that $\chi(\lambda)$ has the following properties:

(i) $\chi_{\bullet}(\lambda) = \lambda$ $\qquad$ there are λ way to color this vertex with λ colors

(ii) $\chi_{G_1 \sqcup G_2}(\lambda) = \chi_{G_1}(\lambda) \cdot \chi_{G_2}(\lambda)$

(iii) $\chi_G = \underbrace{\chi_{G-e}(\lambda)}_{\text{deletion}} - \underbrace{\chi_{G/e}(\lambda)}_{\text{contraction}}$



and so $\overline{\chi}(\lambda)$ has similar properties:

(i) $\overline{\chi}_{\bullet}(\lambda) = \lambda$

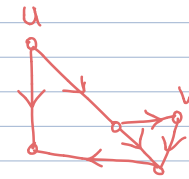
(ii) $\overline{\chi}_{G_1 \sqcup G_2}(\lambda) = \overline{\chi}_{G_1}(\lambda) \cdot \overline{\chi}_{G_2}(\lambda)$

(iii) $\overline{\chi}_G = \underbrace{\overline{\chi}_{G-e}(\lambda)}_{\text{deletion}} + \underbrace{\overline{\chi}_{G/e}(\lambda)}_{\text{contraction}}$

(i) and (ii) follow from the definition
so just need to prove (iii)

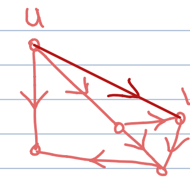
let $e=\{u,v\}$

$\kappa : V(G-e) \to \{1,2,\dots \lambda\}$



$G-e$ with orientation $\mathcal{O}$

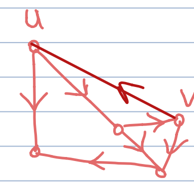
$\mathcal{O}$ is the orientation of $G-e$ that corresponds with κ based on the definition of $\overline{\chi}(\lambda)$ (must be acyclic !!)



G with orientation $\mathcal{O}_1$

$\mathcal{O}_1$ is the orientation of G with $u \to v$

$\mathcal{O}_2$ is the orientation of G with $v \to u$



note κ is defined on $V(G)=V(G-e)$

G with orientation $\mathcal{O}_2$

claim:

for every pair $(\mathcal{O},\kappa)$ either one of $(\mathcal{O}_1,\kappa)$ and $(\mathcal{O}_2,\kappa)$ fits with the definition for $\overline{\chi}(\lambda)$ or both work

only one of $\mathcal{O}_1$ and $\mathcal{O}_2$ work

So $\overline{\chi}(G,\lambda) = [\overline{\chi}(G-e) - \overline{\chi}(G/e)] + 2\cdot\overline{\chi}(G/e)$

$= \overline{\chi}(G-e) + \overline{\chi}(G/e)$

you need to count these twice since both $\mathcal{O}_1$ and $\mathcal{O}_2$ work

proof:

Case 1 : $\kappa(u) > \kappa(v)$

$\mathcal{O}_2$ not compatible $v \to u$ but $\kappa(v) \ngtr \kappa(u)$

$\mathcal{O}_1$ acyclic : assume to get a contradiction the following cycle exists:

$u \to v \to s \to a \to d \to \dots \to u$

but then $\kappa(u) > \kappa(u)$ which is false

Case 2 : $\kappa(v) > \kappa(u)$

similar to case 1 but $\mathcal{O}_2$ is acyclic and $\mathcal{O}_1$ is not compatible

Case 3: $\kappa(u)=\kappa(v)$

both $\mathcal{O}_1$ and $\mathcal{O}_2$ are compatible with condition (2)

at least 1 of $\mathcal{O}_1$ and $\mathcal{O}_2$ is acyclic:

assume to get a contradiction that both have cycles

$\mathcal{O}_1$ $\quad u \to v \to k \to i \to n \to d \to \dots \to u$

$\mathcal{O}_2$ $\quad v \to u \to d \to a \to n \to c \to e \to \dots \to v$

$\mathcal{O}$ $\quad v \to k \to i \to n \to d \to \dots \to u \to d \to a \to n \to c \to e \to \dots \to v$

THIS IS BAD: contradicts that $\mathcal{O}$ is acyclic

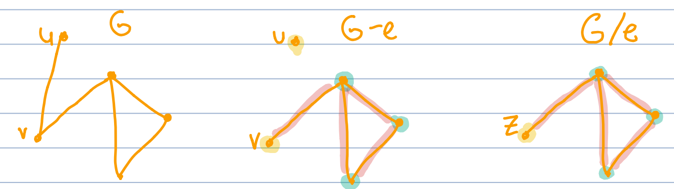
last step in proof: Both $\mathcal{O}_1$ and $\mathcal{O}_2$ are acyclic for $\bar{\chi}(G/e)$ pairs

Make a bijection $\Phi(\kappa, \mathcal{O}) = (\kappa', \mathcal{O}')$

both $\mathcal{O}_1$ and $\mathcal{O}_2$ are acyclic

$\kappa' : G/e \to \{1, 2, \dots, \lambda\}$

$\mathcal{O}'$ is acyclic orientation of G/e and compatible with κ'



$V(G/e) = (V(G-e) - \{u,v\}) \cup \{z\}$

$\chi(G/e) = \chi(G-e)$

let z be the vertex that results from contracting $\{u,v\}$

define the bijection:

$\kappa'(w) = \kappa(w)$ for $w \in V(G/e) - \{u,v\}$

$\kappa'(z) = \kappa(u) = \kappa(v)$

$w_1 \to w_2$ in $\mathcal{O}'$ iff $w_1 \to w_2$ in $\mathcal{O}$

$\bar{\chi}(G,\lambda) = [\bar{\chi}(G-e) - \bar{\chi}(G/e)] + 2 \cdot \bar{\chi}(G/e)$

$= \bar{\chi}(G-e) + \bar{\chi}(G/e)$

Induction time: $\overline{\chi}(\lambda) = (-1)^p \chi(-\lambda)$ for $\lambda \in \mathbb{Z}$ where $\lambda \geq 0$ and $p = \#$ of verticies
(complete)

and so $\overline{\chi}(\lambda)$ has similar properties:

(i) $\overline{\chi}_{\bullet}(\lambda) = \lambda$

(ii) $\overline{\chi}_{G_1 \sqcup G_2}(\lambda) = \overline{\chi}_{G_1}(\lambda) \cdot \overline{\chi}_{G_2}(\lambda)$

(iii) $\overline{\chi}_G = \underbrace{\overline{\chi}_{G-e}(\lambda)}_{\text{deletion}} + \underbrace{\overline{\chi}_{G/e}(\lambda)}_{\text{contraction}}$!!!

Induct on sum of verticies and edges

base case: $\overline{\chi}_{\bullet}(\lambda) = \lambda = (-1)^1 \cdot \chi(-\lambda) = -1 \cdot -\lambda$

note: if you have a bunch of verticies unconnected just apply property 2

assume: that for $p + q \leq k$

$$\overline{\chi}(\lambda) = (-1)^p \chi(-\lambda)$$

$k+1$th case: G is a graph s.t. $p' + q' = k+1$

$$\overline{\chi}(G, \lambda) = \overline{\chi}((G-e), \lambda) + \overline{\chi}(G/e, \lambda)$$

$$= (-1)^{p'} \chi(G-e, -\lambda) + (-1)^{p'-1} \chi(G/e, -\lambda)$$

$$= (-1)^{p'} [\chi(G-e, -\lambda) - \chi(G/e, -\lambda)]$$

$$= (-1)^{p'} \chi(G, -\lambda)$$

note $G-e$ has $q'-1$ edges so verticies + edges $= k$

note G/e has $q'-1$ edges and $p'-1$ verticies so verticies + edges $= k-1$

What was the point of this entire proof?

Stanley's Theorem: $(-1)^p \chi_G(-1) = \#$ acyclic orientations

1.2: $\overline{\chi}(\lambda) = (-1)^p \chi(-\lambda)$ $\Rightarrow$ if $\lambda = 1$ then all the K are the same and compatible with $\mathcal{O}$

$\overline{\chi}(\lambda)$ is equal to the number of pairs $(K, \mathcal{O})$

recall K is any map $K: V \to \{1, 2, \ldots \lambda\}$ (a coloring)

$\mathcal{O}$ is an orientation of G with the following conditions

(1) the orientation $\mathcal{O}$ is acyclic

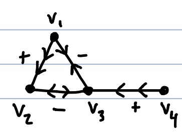
(not redundant any more)

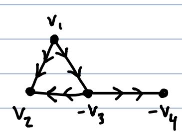
(2) If $u \to v$ in the orientation $\mathcal{O}$, then $K(u) \geq K(v)$

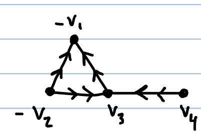
so $\overline{\chi}(1) = \#(K, \mathcal{O})$ pairs $= \# \mathcal{O}$

$= \#$ acyclic orientations

$$\overline{\chi}(1) = (-1)^p \chi(-1)$$


v1
+
-
v2
-
v3
+
v4


v1
v2
-v3
-v4


-v1
-v2
v3
v4